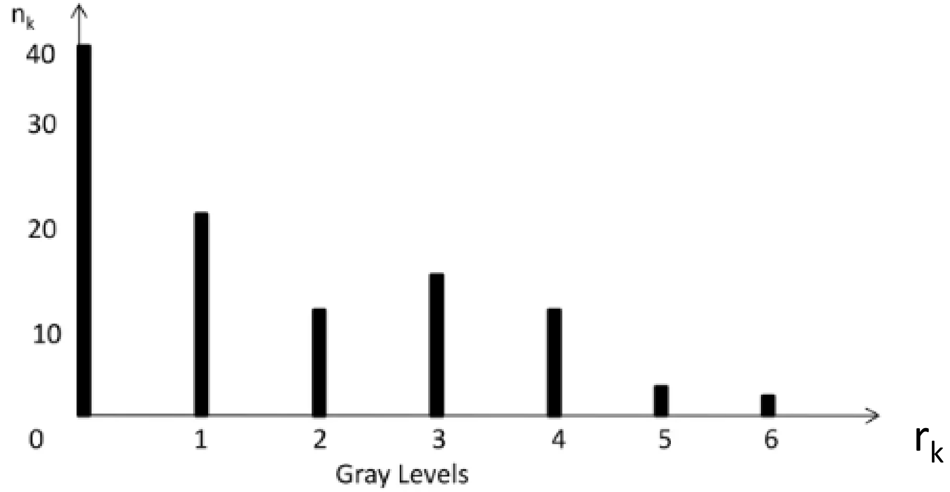
Histogram Processing

Histogram Processing

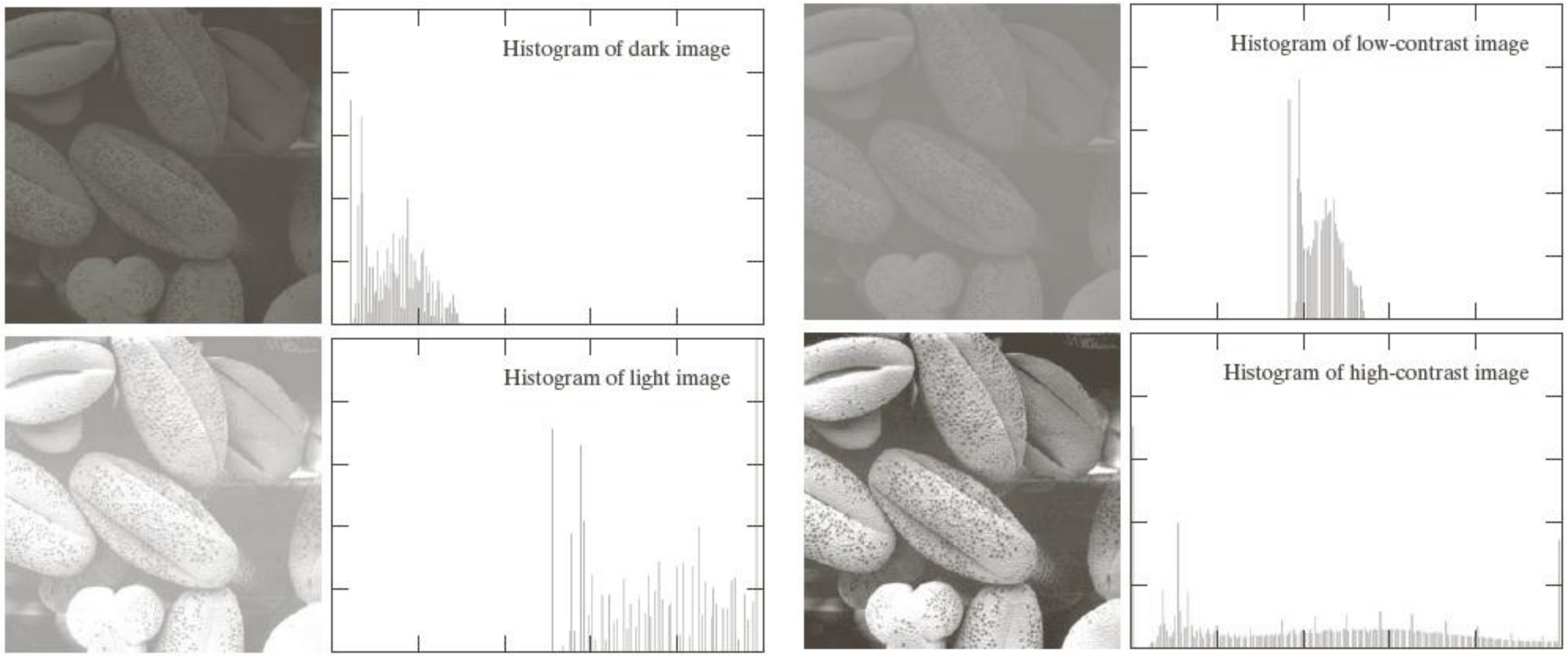
- Histogram Equalization
- Histogram Matching
- Local Histogram Processing
- Using Histogram Statistics for Image Enhancement

Histogram Processing



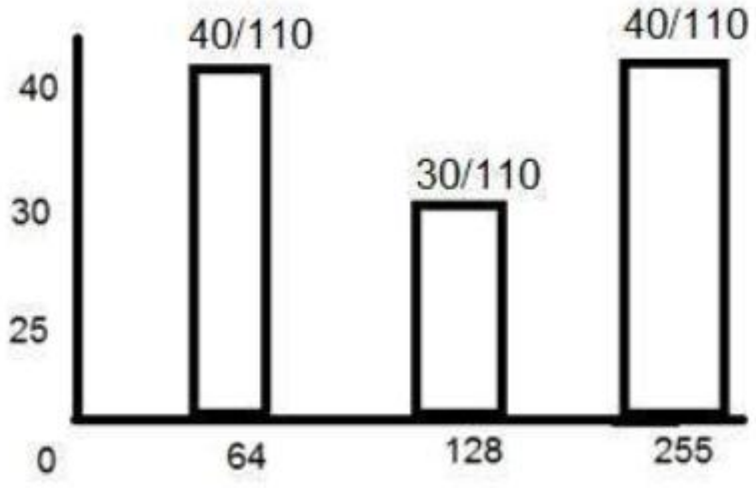
Histogram $h(r_k) = n_k$
r_k is the k^{th} intensity value
n_k is the number of pixels in the image with intensity r_k

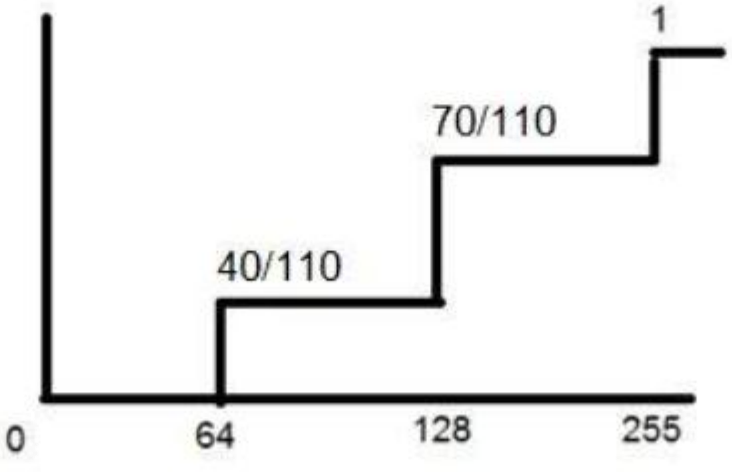
Normalized histogram $p(r_k) = \frac{n_k}{MN}$
n_k: the number of pixels in the image of size $M \times N$ with intensity r_k


Histogram of dark image
Histogram of light image
Histogram of low-contrast image
Histogram of high-contrast image

Histogram Equalization

The intensity levels in an image may be viewed as random variables in the interval [0, L−1].
Let $p_r(r)$ and $p_s(s)$ denote the probability density function (PDF) of random variables r and s.





Histogram Equalization

$$s = T(r) \quad 0 \le r \le L - 1$$

- Assume that

  (a) T(r) is a monotonically increasing function

  (b) $0 \le T(r) \le L - 1$ for $0 \le r \le L - 1$

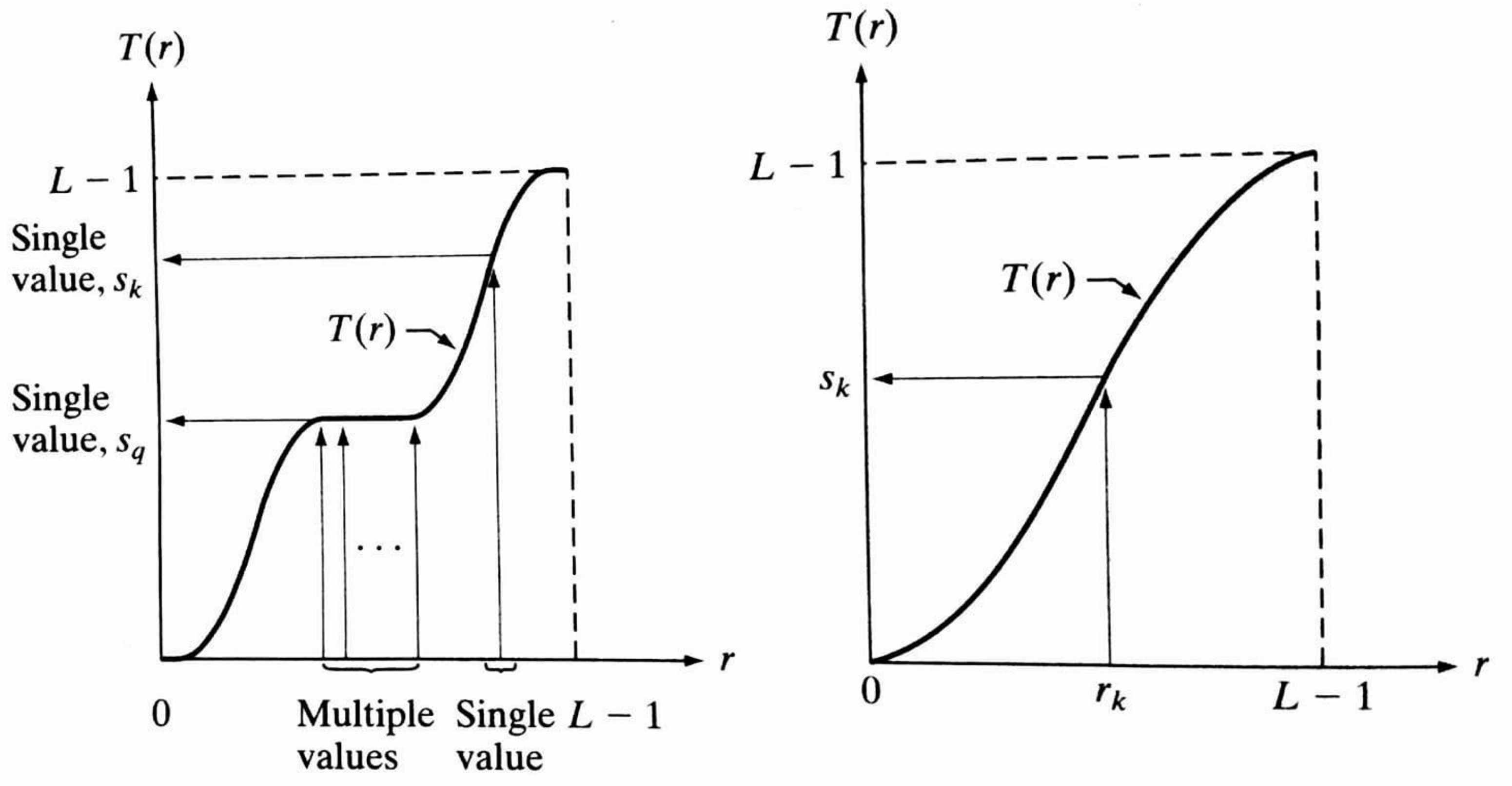
- For the inverse r → s:

$$r = T^{-1}(s) \quad 0 \le s \le L - 1$$

In this case change condition (a) to

(a’) T(r) is a strictly monotonically increasing function

Monotonically increasing and Strictly Monotonically increasing function



a b

FIGURE 3.17
(a) Monotonically increasing function, showing how multiple values can map to a single value.
(b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.

- Let $p_r(r)$ and $p_s(s)$ be a probability density functions. If we assume $p_r(r)$ and $T(r)$ are known, then

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|. \qquad (3.3\text{-}3)$$

Transformation function has the following form where the right side is the cumulative distribution function

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw \qquad (3.3\text{-}4)$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\left[\int_0^r p_r(w)dw\right] = (L-1)p_r(r)$$

$$p_s(s) = p_r(r)\frac{dr}{ds} = p_r(r)\left[\frac{1}{(L-1)p_r(r)}\right] = \frac{1}{L-1}$$

Histogram equalization determine the transformation that seek to produce an output image that has a uniform histogram

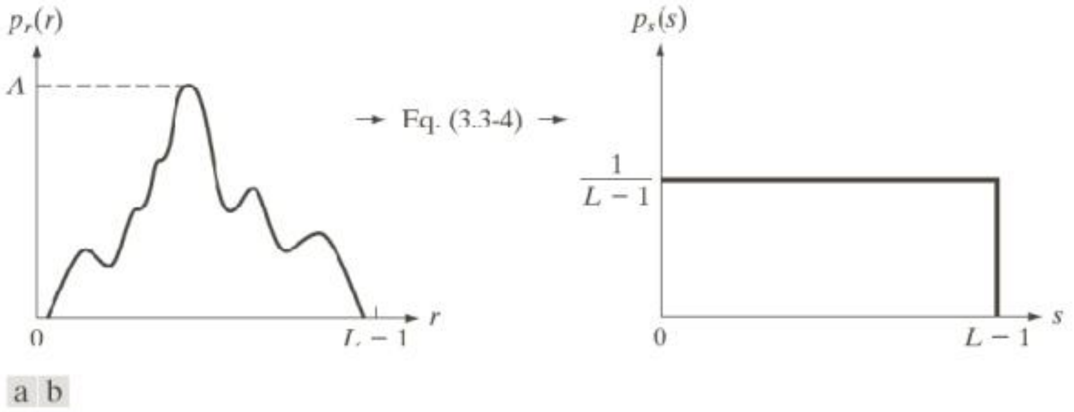


FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

Example

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & ;0 \le r \le L-1 \\ 0 & otherwise \end{cases}$$

$$s = T(r) = (L-1)\int_0^r p_r(w)dw = \frac{2}{L-1}\int_0^r wdw = \frac{r^2}{L-1}$$

$$p_s(s) = p_r(r)\frac{dr}{ds} = \frac{2r}{(L-1)^2}\left|\left[\frac{ds}{dr}\right]^{-1}\right| = \frac{2r}{(L-1)^2}\left|\left[\frac{d}{dr}\frac{r^2}{L-1}\right]^{-1}\right|$$

$$= \frac{2r}{(L-1)^2}\left|\frac{(L-1)}{2r}\right| = \frac{1}{L-1}$$

- For discrete values we deal with probabilities and summation instead of PDF and integrals

$$p_r(r_k) = \frac{n_k}{MN} \qquad k = 0, 1, 2, \ldots, L-1$$

$$\begin{aligned} s_k = T(r_k) &= (L-1)\sum_{j=0}^{k} p_r(r_j) \\ &= \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j \qquad k = 0, 1, 2, \ldots, L-1 \end{aligned} \tag{3.3-8}$$

Example

A 3bit image of size 64x64

[0, L-1] =[0,7]

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33$$

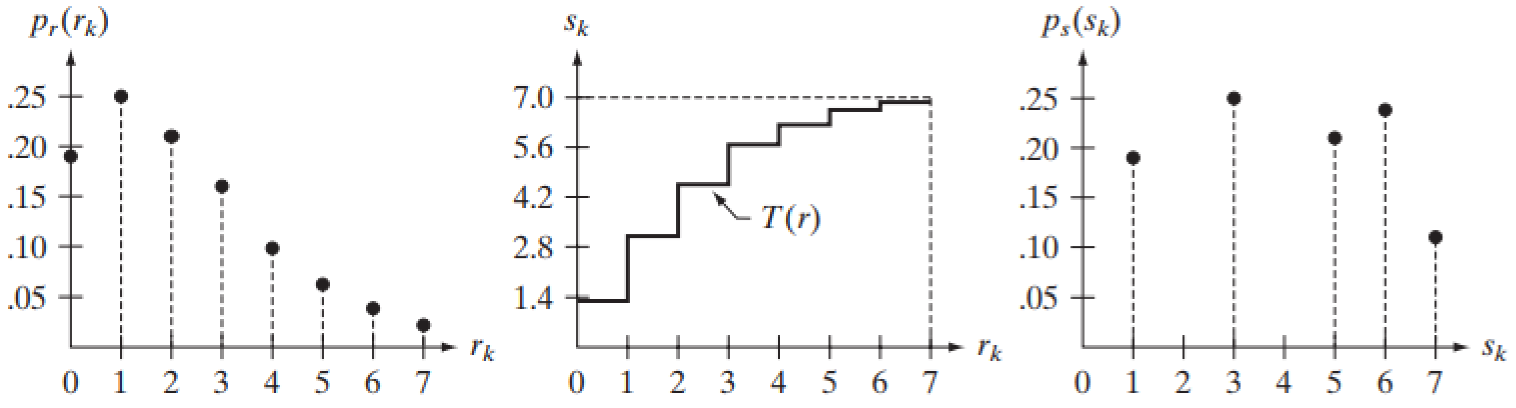
$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

$$s_2 = 4.55, s_3 = 5.67, s_4 = 6.23, s_5 = 6.65, s_6 = 6.86, s_7 = 7.00$$

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

- S0 = 1.33 = 1
- S1 = 3.08 = 3
- S2 = 4.55 = 5
- S3 = 5.67 = 6
- S4 = 6.23 = 6
- S5 = 6.65 = 7
- S6 = 6.86 = 7
- S7 = 7.00 = 7

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

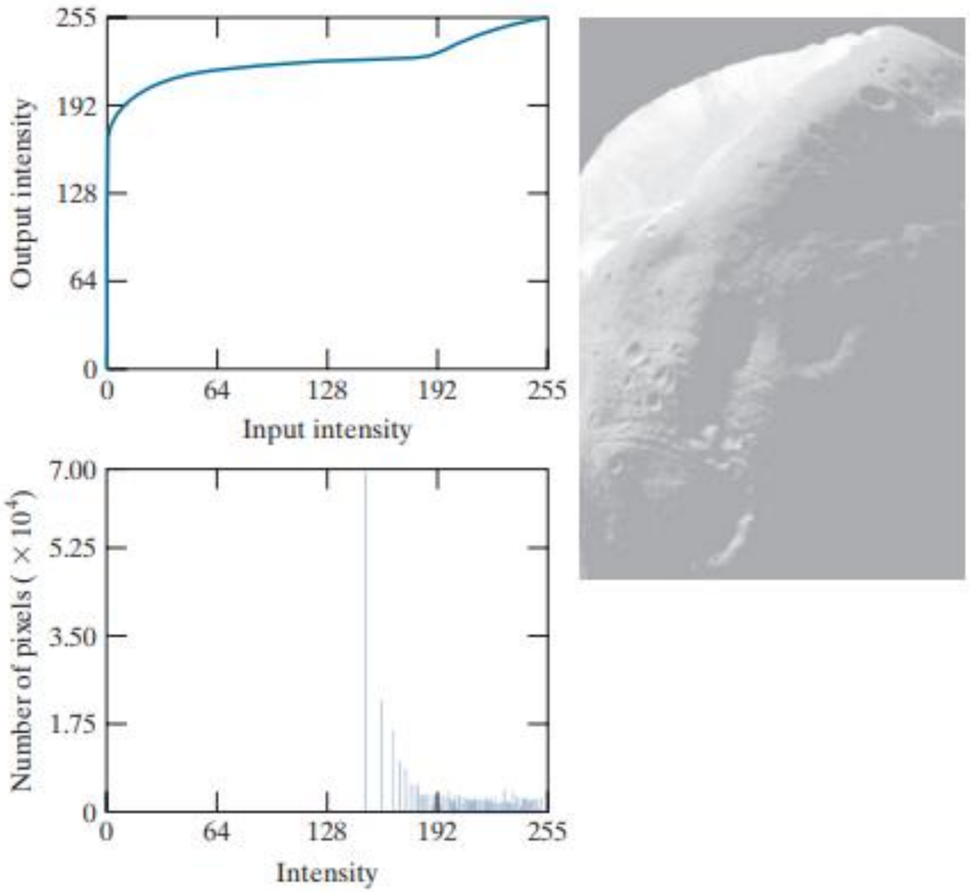


a b c

FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

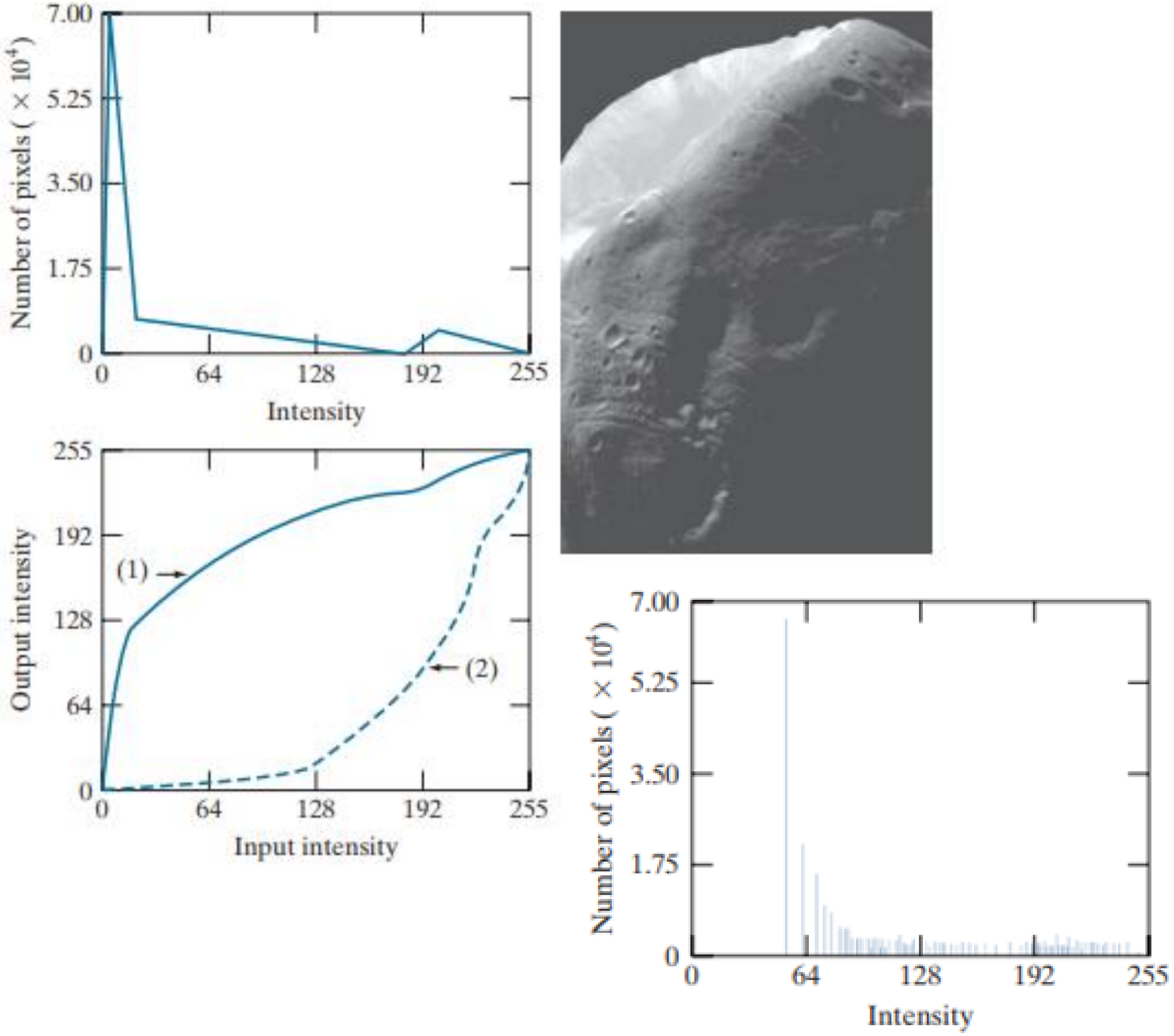
Histogram Matching(Specification)

- Histogram equalization automatically determines a transformation function that seeks to produce an output image that has a uniform histogram
- When automatic enhancement is desired this is good approach because the results are predictable and easy to implement.
- Sometimes it is useful to be able to specify the **shape** of **histogram** that we **wish the processed image to have**.
- So, the method to generate a processed image that has a specified histogram is called **histogram matching** or **histogram specification**.



a b
c

FIGURE 3.24
(a) Histogram equalization transformation obtained using the histogram in Fig. 3.23(b). (b) Histogram equalized image. (c) Histogram of equalized image.



a c
b
d

FIGURE 3.25 Histogram specification. (a) Specified histogram. (b) Transformation $G(z_q)$, labeled (1), and $G^{-1}(s_k)$, labeled (2). (c) Result of histogram specification. (d) Histogram of image (c).

Histogram Matching(Specification)

- Continuous intensities **r** (input image) , z = intensity label of processed image
- $p_z(z)$ is the specified probability density function that we wish the output image to have.

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$G(z) = T(r)$$

$$G(z) = (L-1)\int_0^z p_z(v)\,dv = s$$

$$z = G^{-1}(s) = G^{-1}\left[T(r)\right]$$

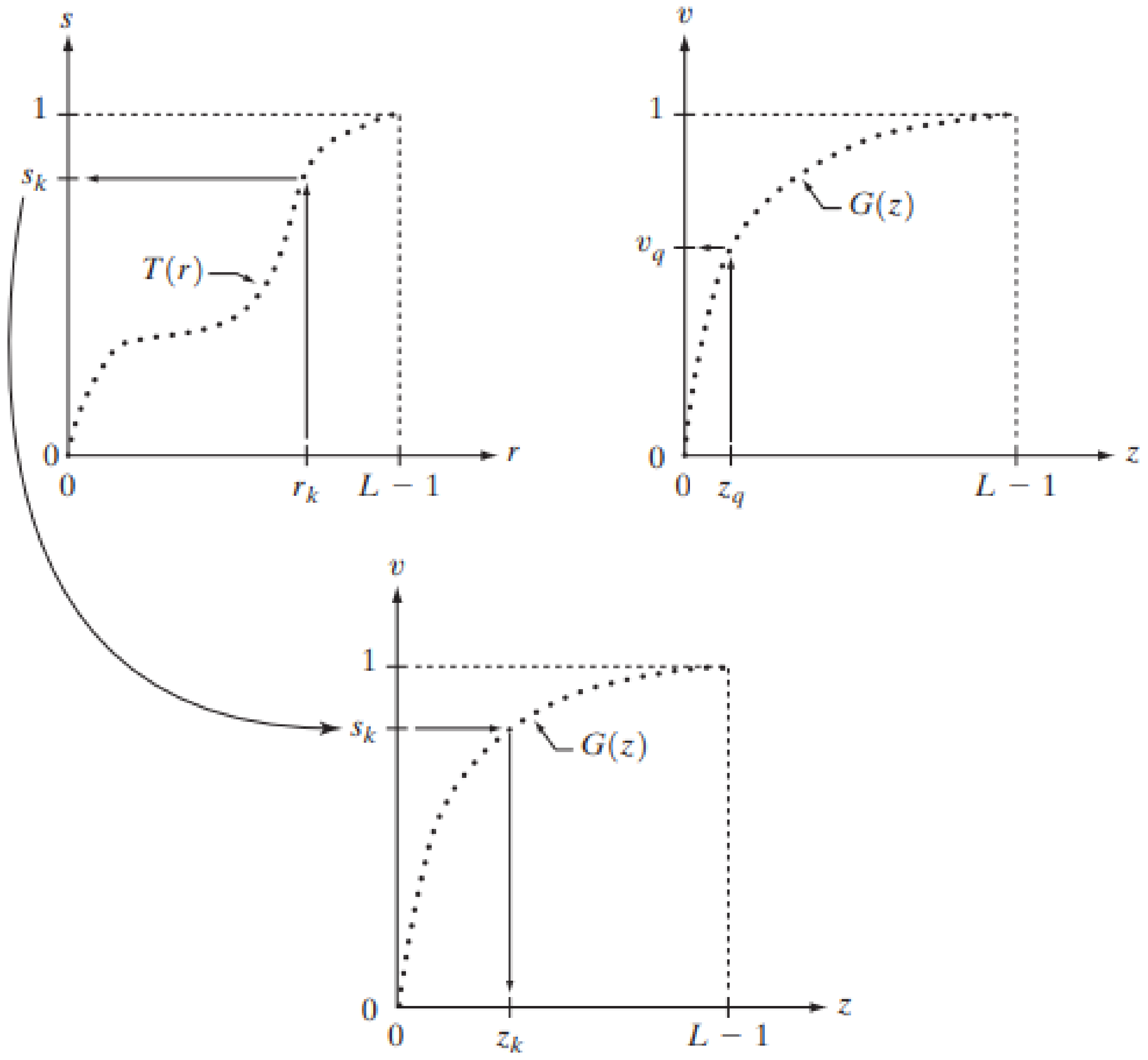


FIGURE 3.19 (a) Graphical interpretation of mapping from r_k to s_k via $T(r)$. (b) Mapping of z_q to its corresponding value v_q via $G(z)$. (c) Inverse mapping from s_k to its corresponding value of z_k.

Histogram Matching(Specification)

- Steps of histogram matching

1. Obtain $p_r(r)$ from the input image and determine the value of s
2. Use a specified PDF to obtain the transformation function G(z)
3. Compute the inverse transformation $z=G^{-1}(s)$
4. Mapping from s to z

Example

Pdf $p_r(r) = \frac{2r}{(L-1)^2}$ for $0 \le r \le L-1$

$p_z(z) = \frac{3z^2}{(L-1)^3}$ for $0 \le z \le L-1$

$$s = T(r) = (L-1)\int_0^r p(w)dw = \frac{2}{L-1}\int_0^r w\,dw = \frac{r^2}{L-1}$$

$$G(z) = (L-1)\int_0^z p(w)dw = \frac{3}{(L-1)^2}\int_0^z w^2\,dw = \frac{z^3}{(L-1)^2}$$

$G(z) = s$,

but $G(z) = z^3/(L-1)^2$;

so $z^3/(L-1)^2 = s$

$$z = [(L-1)^2 s]^{1/3}$$

$$z = [(L-1)^2 s]^{1/3} = [(L-1)^2(\frac{r^2}{L-1})]^{1/3} = [(L-1)\,r^2]^{1/3}$$

Histogram Matching(Specification)

- Discrete

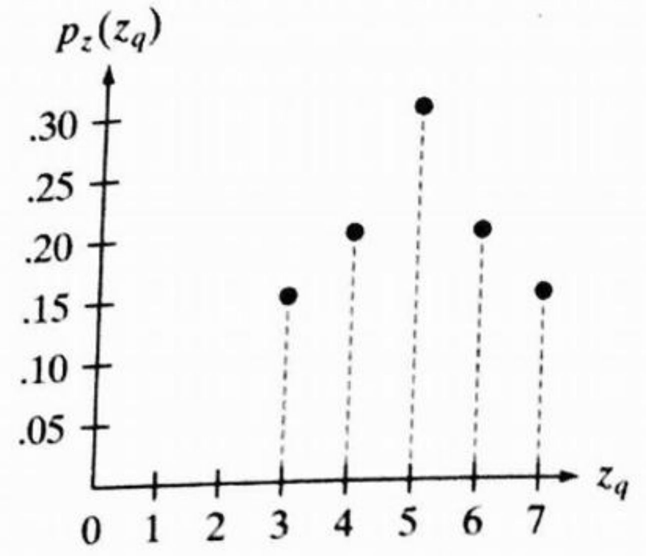
$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) \qquad k = 0,1,2,\ldots,L-1$$

$$G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i)$$

$$G(z_q) = s_k$$

$$z_q = G^{-1}(s_k)$$

z_q	**Specified** $p_z(z_q)$	Equalized $p_z(z_q)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.19
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.25
$z_4 = 4$	0.20	0.00
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11



Histogram Matching(Specification)

$$G(z_0) = 7\sum_{j=0}^{0} p_z(z_j) = 0.00$$

Similarly,

$$G(z_1) = 7\sum_{j=0}^{1} p_z(z_j) = 7\left[p(z_0) + p(z_1)\right] = 0.00$$

and

$$G(z_2) = 0.00 \quad G(z_4) = 2.45 \quad G(z_6) = 5.95$$

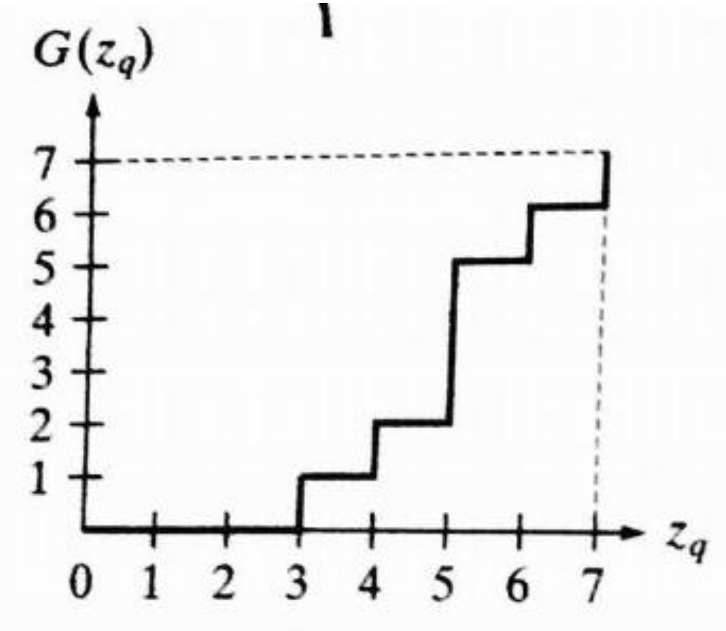
$$G(z_3) = 1.05 \quad G(z_5) = 4.55 \quad G(z_7) = 7.00$$

Histogram Matching(Specification)

$G(z_0) = 0.00 \rightarrow 0$
$G(z_1) = 0.00 \rightarrow 0$
$G(z_2) = 0.00 \rightarrow 0$
$G(z_3) = 1.05 \rightarrow 1$

$G(z_4) = 2.45 \rightarrow 2$
$G(z_5) = 4.55 \rightarrow 5$
$G(z_6) = 5.95 \rightarrow 6$
$G(z_7) = 7.00 \rightarrow 7$

z_q	$G(z_q)$
$z_0 = 0$	0
$z_1 = 1$	0
$z_2 = 2$	0
$z_3 = 3$	1
$z_4 = 4$	2
$z_5 = 5$	5
$z_6 = 6$	6
$z_7 = 7$	7



Histogram Matching(Specification)

- S0 = 1.33 = 1
- S1 = 3.08 = 3
- S2 = 4.55 = 5
- S3 = 5.67 = 6
- S4 = 6.23 = 6
- S5 = 6.65 = 7
- S6 = 6.86 = 7
- S7 = 7.00 = 7

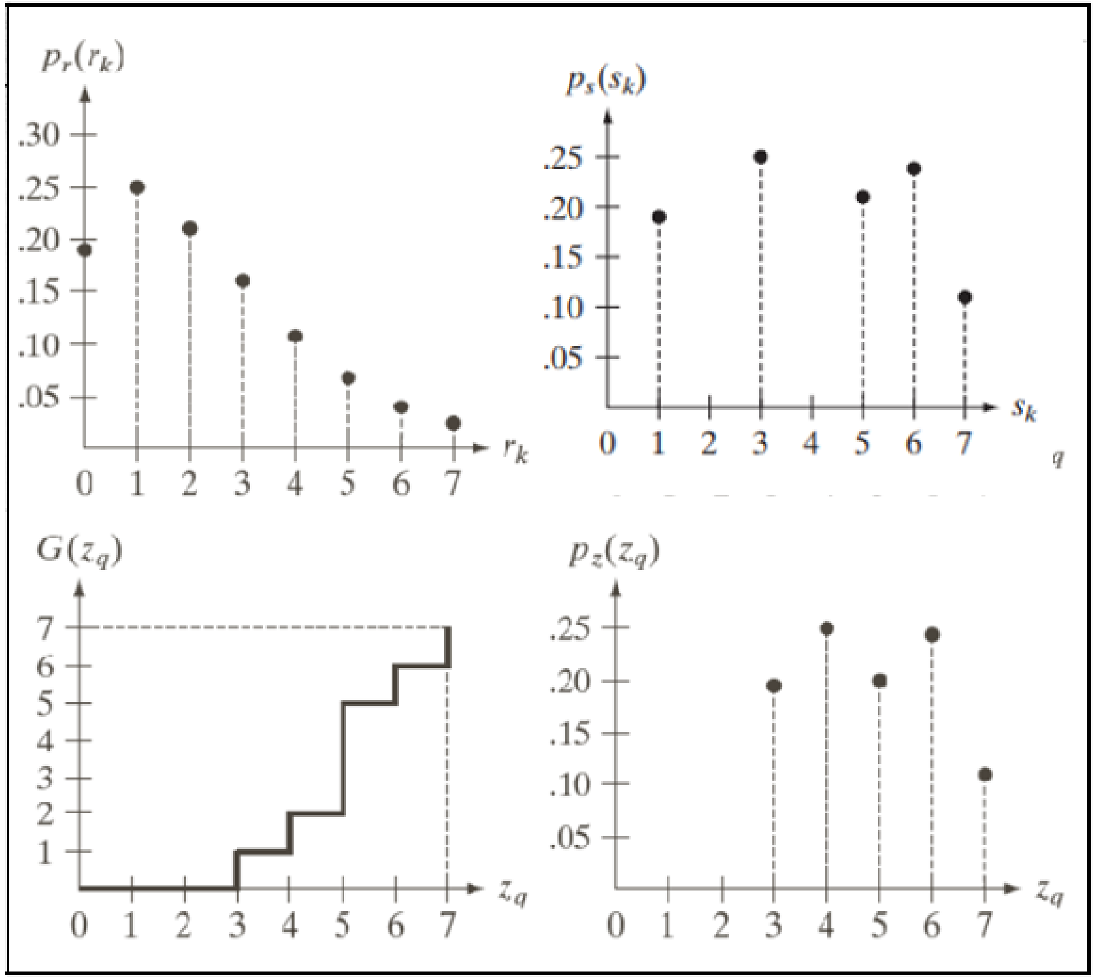
z_q	$G(z_q)$
$z_0 = 0$	0
$z_1 = 1$	0
$z_2 = 2$	0
$z_3 = 3$	1
$z_4 = 4$	2
$z_5 = 5$	5
$z_6 = 6$	6
$z_7 = 7$	7

s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7

Histogram Matching(Specification)

s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7

z_q	Equalized $p_z(z_q)$
$z_0 = 0$	0.00
$z_1 = 1$	0.19
$z_2 = 2$	0.00
$z_3 = 3$	0.25
$z_4 = 4$	0.00
$z_5 = 5$	0.21
$z_6 = 6$	0.24
$z_7 = 7$	0.11

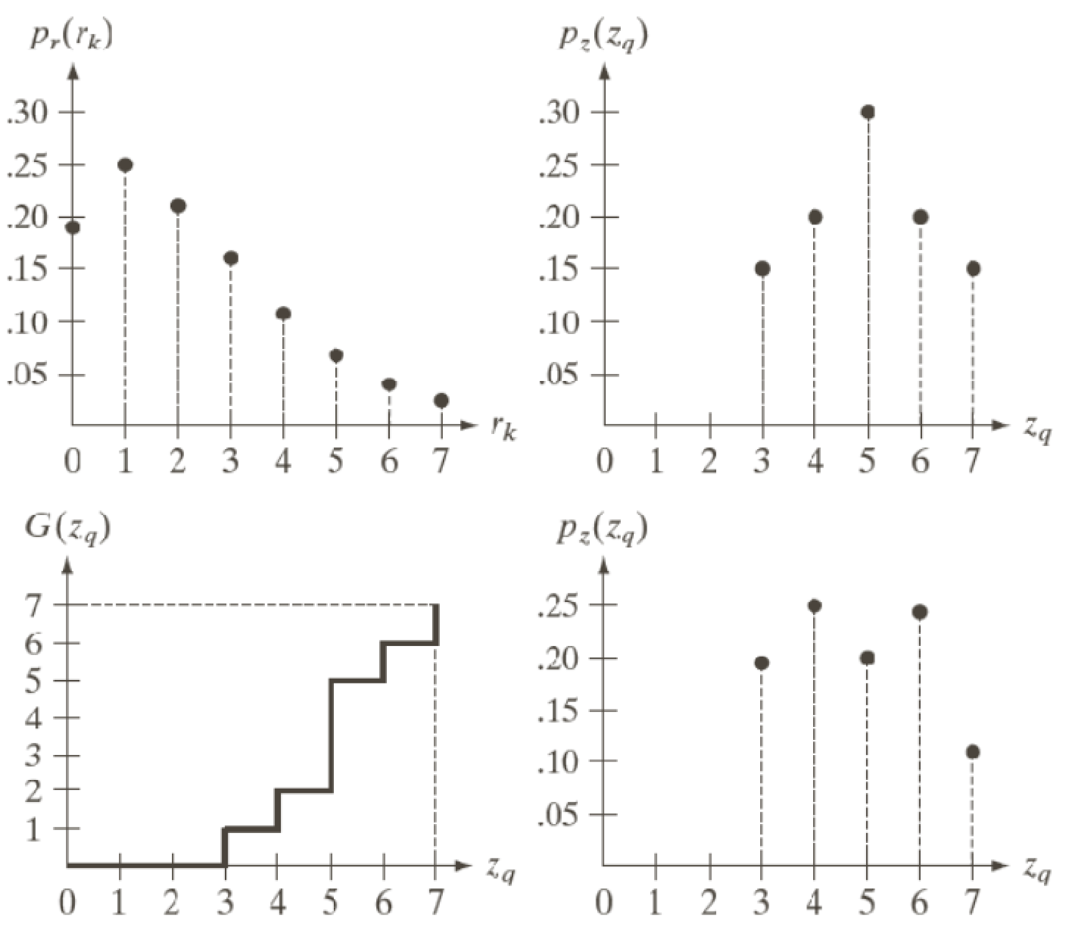


Histogram Matching(Specification)

- S0 = 1.33 = 1
- S1 = 3.08 = 3
- S2 = 4.55 = 5
- S3 = 5.67 = 6
- S4 = 6.23 = 6
- S5 = 6.65 = 7
- S6 = 6.86 = 7
- S7 = 7.00 = 7

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7



Histogram Matching(Specification)

z_q	Specified $p_z(z_q)$	Actual $p_z(z_q)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.00
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.19
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11

Local Histogram Processing

- The histogram processing methods discussed in the previous two sections are *global*, in the sense that pixels are modified by a transformation function based on the intensity distribution of an entire image.

- For local histogram processing, The procedure is to define a neighborhood and move its center from pixel to pixel.

- At each location, the histogram of the points in the neighborhood is computed and either a histogram equalization or histogram specification transformation function is obtained.

Local Histogram Processing

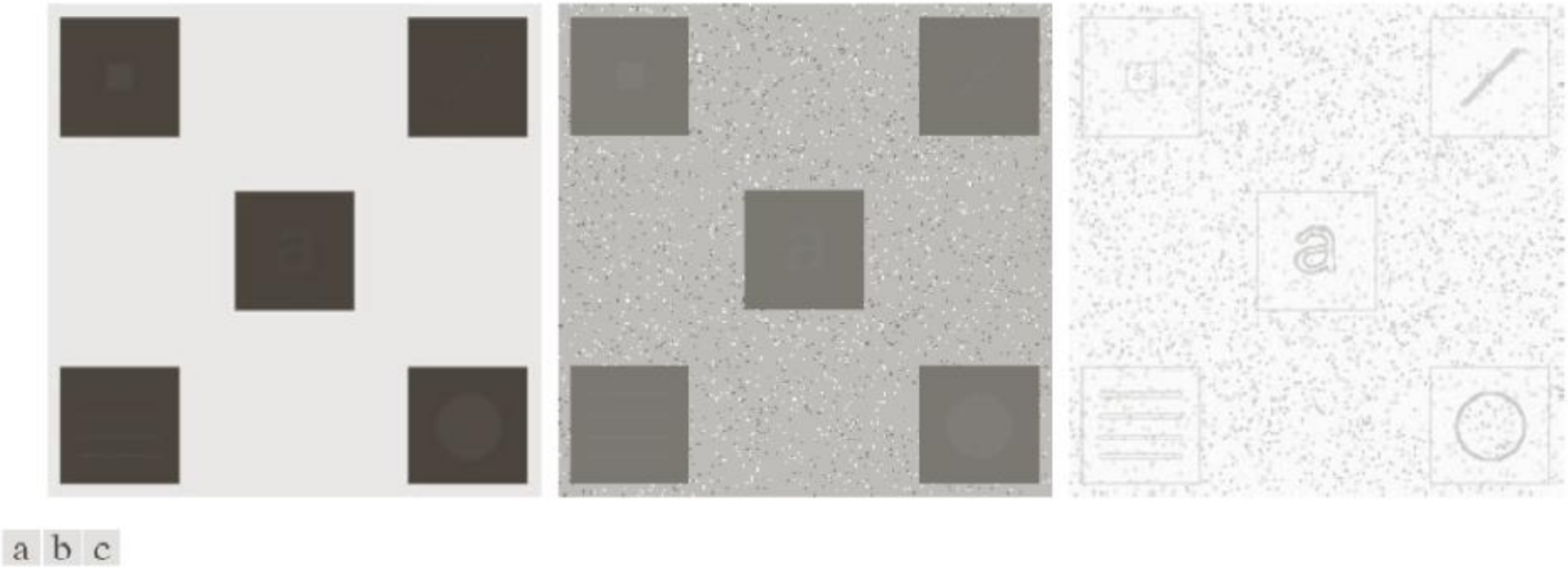


FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3 × 3.

Using Histogram Statistics for Image Enhancement

- Let *r* denote a discrete random variable representing intensity values in the range and let $p(r_i)$ denote the normalized histogram [0, L - 1], component corresponding to value r_i
- $p(r_i)$ is an estimate of the probability that intensity r_i occurs in the image from which the histogram was obtained.
- the *n*th moment of *r* about its mean is defined as

$$\mu_n(r) = \sum_{i=0}^{L-1} (r_i - m)^n p(r_i)$$

- where *m* is the mean (average intensity) value of *r* (i.e., the average intensity of the pixels in the image):

$$m = \sum_{i=0}^{L-1} r_i p(r_i)$$

- The second moment is particularly important:

$$\mu_2(r) = \sum_{i=0}^{L-1} (r_i - m)^2 p(r_i)$$

- When working with only the mean and variance, it is common practice to estimate them directly from the sample values, without computing the histogram.
- They are given by the following familiar expressions from basic statistics:

$$m = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

$$\sigma^2 = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[f(x, y) - m \right]^2$$

- Consider the following 2-bit image of size 5 X 5:

$$p(r_0) = \frac{6}{25} = 0.24; \; p(r_1) = \frac{7}{25} = 0.28;$$

$$p(r_2) = \frac{7}{25} = 0.28; \; p(r_3) = \frac{5}{25} = 0.20$$

$$\begin{aligned} m &= \sum_{i=0}^{3} r_i \, p(r_i) \\ &= (0)(0.24) + (1)(0.28) + (2)(0.28) + (3)(0.20) \\ &= 1.44 \end{aligned}$$

0	0	1	1	2
1	2	3	0	1
3	3	2	2	0
2	3	1	0	0
1	1	3	2	2

Letting f(x,y) denote the preceding 5x5 array

$$m = \frac{1}{25}\sum_{x=0}^{4}\sum_{y=0}^{4} f(x, y)$$

$$= 1.44$$

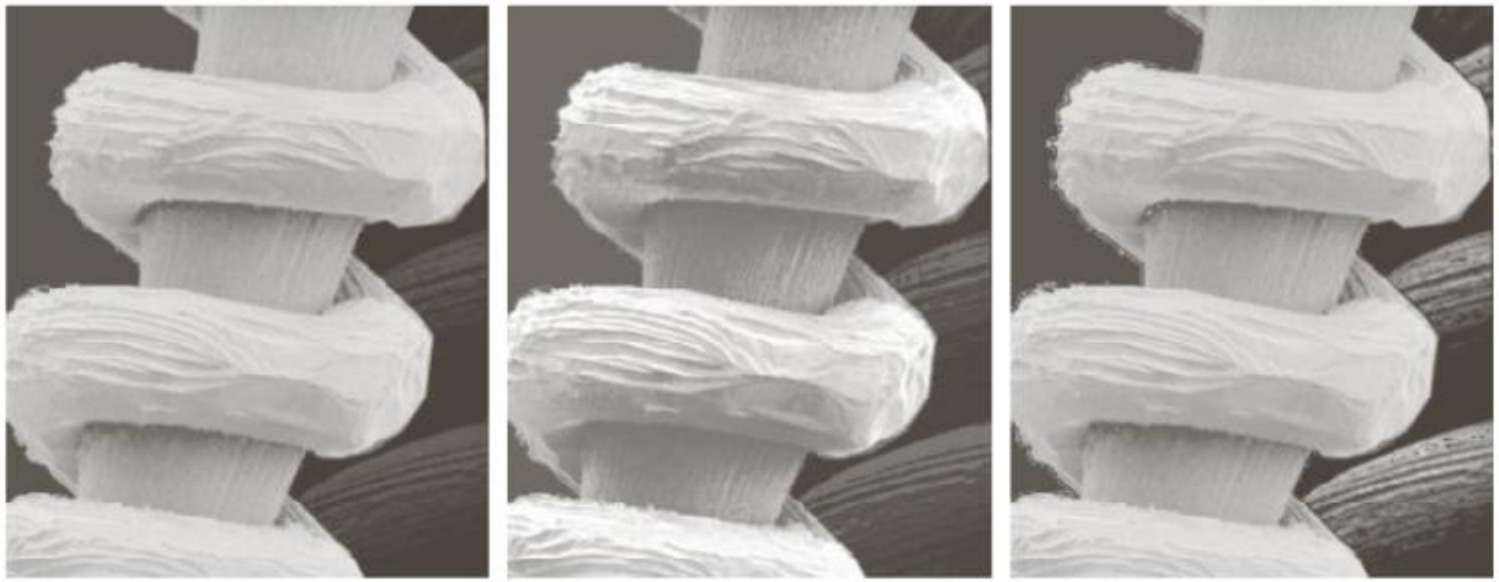
A more powerful use of these parameters is in local enhancement, where **the *local* mean and variance** are used as the basis for making changes that depend on image characteristics in a neighborhood about each pixel in an image

- Let S_{xy} denote a neighborhood (subimage) of specified size, centered on (*x*, *y*). The mean value of the pixels and the variance of the pixels in this neighborhood is given by the expression

$$m_{S_{xy}} = \sum_{i=0}^{L-1} r_i p_{S_{xy}}(r_i)$$

$$\sigma^2_{S_{xy}} = \sum_{i=0}^{L-1} (r_i - m_{S_{xy}})^2 p_{S_{xy}}(r_i)$$

- In this particular case, the problem is to enhance dark areas while leaving the light area as unchanged as possible because it does not require enhancement.



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

- Formulate an enhancement method that can tell the difference between dark and light and, at the same time, is capable of enhancing only the dark areas.

- The preceding approach as follows. Let represent the value of an image at any image coordinates (*x*, *y*), and let represent the corresponding enhanced value at those coordinates. Then

$$g(x, y) = \begin{cases} E \cdot f(x, y) & \text{if } m_{S_{xy}} \le k_0 m_G \text{ AND } k_1 \sigma_G \le \sigma_{S_{xy}} \le k_2 \sigma_G \\ f(x, y) & \text{otherwise} \end{cases}$$

$$g(x, y) = \begin{cases} E \cdot f(x, y) & \text{if } m_{S_{xy}} \leq k_0 m_G \text{ AND } k_1 \sigma_G \leq \sigma_{S_{xy}} \leq k_2 \sigma_G \\ f(x, y) & \text{otherwise} \end{cases}$$

- E = 4.0, k0 = 0.4, k1 = 0.02 , $k_{2} =$ 0.4
- The value of was chosen as less than half the global mean because we can see by looking at the image that the areas that require enhancement definitely are dark enough to be below half the global mean.

- https://www.tutorialspoint.com/dip/histogram_equalization.htm